UDC 521.13

ON STABILITY OF CONSTANT LAPLACE SOLUTIONS OF THE UNRESTRICTED THREE-BODY PROBLEM

PMM Vol. 42, No. 6, 1978, pp. 1026–1032

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(Received May 11, 1978)

It is shown that in the absence of third order resonances [1] Laplace solutions retain stability in the second order within the limits of the Routh –Joukowski necessary conditions of stability. When third order resonances and their interaction take place in a system, the question of stability in the second order and that of Liapunov instability is completely solved by the present investigation in conjunction with that in [2].

1. We derive the equations of perturbed motion for this problem on the basis of Liapunov's investigations, taking into account terms of up to the second order of smallness in the right-hand sides of these.

Let three mass points M_0, M_1, and M_2 be attracted to each other in accordance with the law

$$F_{ij} = fM_iM_jr_{ij}^n \quad (i, j = 0, 1, 2;\ i \neq j) \tag{1.1}$$

where f is a constant, M_i and M_j are masses of points, r_{ij} are distances between these, and n is a real number.

Conforming to Liapunov's method we introduce a moving coordinate system with origin at point M_0. The axes of abscissas and ordinates are, respectively, $M_0\xi$ which runs from the origin M_0 to point M_1, and the line $M_0\eta$ normal to the latter. They lie in the plane of triangle $(M_0M_1M_2)$ and are directed so that each is at an acute angle to line M_0M_2. The third axis $M_0\zeta$ is normal to the plane of triangle $(M_0M_1M_2)$ and directed so that the system $M_0\xi\eta\zeta$ is right-handed.

Liapunov obtained differential equations of motion for the unrestricted three-body problem using variables $r_1, r_2, \psi, \omega_1, \omega_2$, and ω_3, where $r_1 = r_{01}, r_2 = r_{02}, \psi$ is the angle between directions M_0M_1 and M_0M_2, and ω_1, ω_2, and ω_3 are projections of angular velocity of the moving coordinate system $M_0\xi\eta\zeta$ on axes $M_0\xi$, $M_0\eta$, and $M_0\zeta$, respectively [3]. These equations have particular solutions when the three points M_0, M_1, and M_2 moving in an invariant plane form an equilateral triangle (solution of Laplace [3]) or lie on a single straight line.

We shall consider only the constant Laplace solutions when

$$r_1 = \rho,\ r_2 = \rho,\ \psi = \pi/3,\ \omega_1 = 0,\ \omega_2 = 0,\ \omega_3 = \omega \tag{1.2}$$
$$(\rho\omega^2 = f(M_0 + M_1 + M_2)\rho^n,\ \rho^2\omega = c_*)$$

(constants ρ and ω are linked by the relations shown in parentheses and c_* is an arbitrary constant).

We set

$$r_1 = \rho\,(1 + \xi), \quad r_2 = \rho\,(1 + \xi + x), \quad \psi = \pi/3 + y \qquad (1.3)$$
$$\omega_3 = \omega \quad (1 + \eta)$$

and assume that the quantities x, y, ξ, η, ω_1, and ω_2 and their derivatives are small of the same order [3].

In the case of unperturbed motion (1.2) we evidently have

$$x = 0,\ y = 0,\ \xi = 0,\ \eta = 0,\ \omega_1 = 0,\ \omega_2 = 0 \qquad (1.4)$$

hence the problem of stability of the particular solution (1.2) reduces to that of Liapunov stability of the zero solution (1.4) of differential equations derived from Liapunov's equations for the unrestricted three-body problem [3] after the substitution of (1.3).

We effect this substitution by expanding the nonlinear terms of equations in series in powers of perturbations (1.4). After necessary transformations, omitted here owing to their unwieldiness, we obtain a system of equations of perturbed motion of the form

$$d\eta_*/d\theta = 0 \qquad (1.5)$$
$$\frac{d\xi}{d\theta} = \xi_1, \qquad \frac{d\xi_1}{d\theta} = 2\eta_* - (n+3)\,\xi + \left[\frac{3}{4}(1-n)\,m_2 - 2\alpha_m\right] x +$$
$$\left[\frac{\sqrt{3}}{4}(1-n)\,m_2 - 2\beta_m\right] y + \beta_m x_1 - \alpha_m y_1 + \Phi_1 + \ldots$$
$$\frac{dx}{d\theta} = x_1, \qquad \frac{dx_1}{d\theta} = 2y_1 + (1-n)\,[(1-\mu)\,x + \mu' y] + \Phi_2 + \ldots$$
$$\frac{dy}{d\theta} = y_1, \qquad \frac{dy_1}{d\theta} = -2x_1 + (1-n)\,[\mu' x + \mu y] + \Phi_3 + \ldots$$
$$\frac{d\Omega_1}{d\theta} = -\Omega_2 + \Phi_4 + \ldots, \qquad \frac{d\Omega_2}{d\theta} = \Omega_1 + \Phi_5 + \ldots$$

where

$$\Phi_1 = \frac{n\,(1-n)}{2}\,\xi^2 + (1-n)\,m_2\left(\frac{5n-3}{16}\,x^2 + P_-\right) + \Omega_2^2 + \eta^2 + 2\xi\eta +$$
$$2\left(\eta - \eta_* + 2\xi + \alpha_m x + \beta_m y - \frac{\beta_m}{2}\,x_1 + \frac{\alpha_m}{2}\,y_1\right)$$
$$\Phi_2 = \frac{n\,(1-n)}{2}\,(m_0 + m_2)\,(x^2 + 2\xi x) + (1-n)\,m_2\left(\frac{n+9}{16}\,x^2 + P_+\right) -$$
$$(1-n)\,m_2\left(\frac{5n-3}{16}\,x^2 + P_-\right) + 2x\eta + y_1^2 + 2\,(\xi + x + \eta)\,y_1 +$$
$$\frac{3}{4}\,\Omega_1^2 - \frac{\sqrt{3}}{4}\,\Omega_1\Omega_2 - \frac{3}{4}\,\Omega_2^2$$
$$\Phi_3 = (1-n)\,m_1\left(\frac{\sqrt{3}\,(n+1)}{16}\,x^2 + Q_-\right) +$$
$$(1-n)\,m_2\left(-\frac{\sqrt{3}\,(3n+5)}{16}\,x^2 + Q_+\right) + 2\,(\xi + x)\,x_1 - 2x_1\eta -$$
$$2\,(\xi_1 + x_1)\,y_1 - (1-n)\,(\mu' x + \mu y)\,(\xi + x) -$$
$$2x\left[-\xi_1 + (1-n)\,m_2\left(\frac{\sqrt{3}}{4}\,x - \frac{3}{4}\,y\right)\right] + \frac{\sqrt{3}}{4}\,\Omega_1^2 + \frac{3}{4}\,\Omega_1\Omega_2 -$$

$$\frac{\sqrt{3}}{4}\Omega_2^2$$

$$\Phi_4 = -\eta\Omega_2 - \frac{2}{\sqrt{3}}(\Omega_1 + \sqrt{3}\Omega_2)y_1 + \frac{2}{\sqrt{3}}(-\sqrt{3}\Omega_1 + \Omega_2)x_1 - 2\Omega_1\xi_1, \quad \Phi_5 = \eta\Omega_1 - 2\Omega_2\xi_1$$

$$P_\pm = \frac{3n+5}{16}y^2 + \frac{3(n-1)}{4}\xi x + \frac{\sqrt{3}n}{4}\xi y + \frac{\sqrt{3}(n\pm 5)}{8}xy$$

$$Q_\pm = \frac{3\sqrt{3}(n-1)}{16}y^2 \mp \frac{\sqrt{3}(n-1)}{4}\xi x + \frac{3n}{4}\xi y + \frac{3n\pm 1}{8}xy$$

$$\Omega_1 = \frac{\omega_1}{\omega}, \quad \Omega_2 = \frac{\omega_2}{\omega}, \quad d\theta = \omega\, dt$$

$$m_i = \frac{M_i}{M_0 + M_1 + M_2} \quad (i = 0, 1, 2), \quad \mu = \frac{3}{4}(m_1 + m_2)$$

$$\mu' = \frac{\sqrt{3}}{4}(m_1 - m_2)$$

$$\alpha_m = \frac{(2M_0 + M_1)M_2}{M_0M_1 + M_0M_2 + M_1M_2}, \quad \beta_m = \frac{\sqrt{3}M_1M_2}{M_0M_1 + M_0M_2 + M_1M_2}$$

where terms of order higher than the second with respect to variables are omitted, t is the time, and η_* is the arbitrary constant of the energy integral which is defined as follows:

$$\eta = \eta_* - \Big\{R + \frac{n+1}{2}\xi^2 + \frac{[4(n+1)m_0 + (n+7)m_1]m_2}{8\nu}x^2 + \frac{3n-1}{8\nu}m_1m_2y^2 + \frac{\xi_1^2}{2} + \frac{(m_0+m_1)m_2}{2\nu}(x_1^2 + y_1^2) + \frac{n+1}{2}\alpha_m\xi x + \frac{n+3}{2}\beta_m\xi y + \frac{\beta_m}{2}yy_1 + \frac{\alpha_m}{2}\xi_1x_1 + \frac{\beta_m}{2}\xi_1y_1 - \frac{\beta_m}{2}\xi x_1 + \alpha_m\xi y_1 + \frac{\beta m}{2}x\xi_1\frac{n+3}{4}\beta_m xy - \frac{m_1m_2}{2\nu}yx_1 + \frac{(m_0+m_1)m_2}{2\nu} \times \left(\frac{3}{4}\Omega_1^2 - \frac{\sqrt{3}}{2}\Omega_1\Omega_2 + \frac{1}{4}\Omega_2^2\right) + \frac{m_1m_2}{2\nu}(\sqrt{3}\Omega_1 - \Omega_2)\Omega_2\Big\} + \frac{1}{2}R^2 - \frac{1}{2}R\eta_* + \ldots$$

$$R = 2\xi + \alpha_m x + \beta_m y - \frac{\beta_m}{2}x_1 + \frac{\alpha_m}{2}y_1, \quad \nu = m_0m_1 + m_0m_2 + m_1m_2$$

The characteristic equation of system (1.5) has one zero and four pairs of pure imaginary roots [1]

$$\varkappa_s = \pm i\lambda_s \ (s = 1, \ldots, 4)$$

$$\lambda_1 = 1, \quad \lambda_2 = (n+3)^{1/2}, \quad \lambda_{3,4} = \left[\frac{n+3}{2}(1 \pm \sqrt{1-\chi})\right]^{1/2}$$

$$\chi = 3\left(\frac{n-1}{n+3}\right)^2\nu$$

It was shown in [1] that within the limits of necessary conditions of the Routh — Joukowski stability there are eight third order resonances

$$\lambda_1 = \lambda_3 \pm \lambda_4,\ \lambda_1 = 2\lambda_2,\ \lambda_1 = 2\lambda_3,\ \lambda_1 = 2\lambda_4 \qquad (1.6)$$
$$\lambda_4 = 2\lambda_1,\ \lambda_2 = 2\lambda_4,\ \lambda_3 = 2\lambda_4$$

and that for $n = -2$ and $\nu = 1/36$ interaction of resonances $\lambda_1 = \lambda_2 = 2\lambda_4$ occurs [2].

Stability of resonance curves (1.6) was investigated in [2] in a nonlinear formulation with allowance only for perturbations for which $\eta_* = 0$.

2. It should be noted that in linear approximation system (1.5) decomposes into two second order systems (in ξ, ξ_1 and Ω_1, Ω_2), a fourth order system (in x, x_1, y, y_1) and an equation for η_*. Hence after transformations

$$\xi = \frac{\zeta}{\lambda_2} + \frac{2\eta_*}{\lambda_2} - \frac{1}{2}(\alpha_m x + \beta_m y), \qquad \xi_1 = \zeta_1 - \frac{1}{2}(\alpha_m x_1 + \beta_m y_1)$$

$$z_1 = \Omega_1 + i\Omega_2,\ \bar z_1 = \Omega_1 - i\Omega_2,\ z_2 = \zeta_1 + i\zeta,\ \bar z_2 = \zeta_1 - i\zeta$$

$$(x, x_1, y, y_1) = \left\| \begin{matrix} -\frac{i}{\lambda_3} a_3^- & -\frac{i}{\lambda_4} a_4^- & \frac{i}{\lambda_3} a_3^+ & \frac{i}{\lambda_4} a_4^+ \\ a_3^- & a_4^- & a_3^+ & a_4^+ \\ -\frac{i}{\lambda_3} & -\frac{i}{\lambda_4} & \frac{i}{\lambda_3} & \frac{i}{\lambda_4} \\ 1 & 1 & 1 & 1 \end{matrix} \right\| \left\| \begin{matrix} z_3 \\ z_4 \\ \bar z_3 \\ \bar z_4 \end{matrix} \right\|$$

$$a_j^\pm = [-(1-n) \pm 2\lambda_j i]/L_j,\ L_j = (1-n)(1-\mu) + \lambda_j^2$$
$$(j = 3, 4)$$

where z_s, $\bar z_s$ $(s = 1, \ldots, 4)$ are complex conjugate variables, system (1.5) assumes the form

$$\eta_*^\cdot = 0 \qquad (2.1)$$
$$z_s^\cdot = i\lambda_s z_s + \eta_* \sum_{j=1}^{4} (A_{sj} z_j + \bar B_{sj} \bar z_j) + \sum C_s^* \prod_{j=1}^{4} z_j^{k_{sj}} \bar z_j^{l_{sj}} + \ldots$$
$$\bar z_s^\cdot = -i\lambda \bar z_{ss} + \eta_* \sum_{j=1}^{4} (\bar A_{sj} \bar z_j + B_{sj} z_j) + \sum \bar C_s^* \prod_{j=1}^{4} \bar z_j^{k_{sj}} z_j^{l_{sj}} + \ldots$$
$$(s = 1, \ldots, 4)$$

where A_{sj}, $\bar A_{sj}$, B_{sj}, $\bar B_{sj}$, C_s^*, and $\bar C_s^*$ are complex conjugate coefficients of quandratic terms, the asterisk replaces the superscripts $(k_{s1}, \ldots, k_{s4}, l_{s1}, \ldots, l_{s4})$, and summation is carried out over all nonnegative integral numbers $k_{s1}, \ldots, k_{s4}$ and $l_{s1}, \ldots, l_{s4}$ whose sum is equal two.

Let in system (2.1) $\operatorname{Re} A_{ss} = 0$ $(s = 1, \ldots, 4)$. The following theorems are then valid.

T h e o r e m 1. Let a resonance of the form $\lambda_\alpha = 2\lambda_\beta$ be present in system (2.1). Let us consider the terms $C_\alpha^* z_\beta^2$ and $C_\beta^* z_\alpha \bar z_\beta$ in the right-hand sides of respective equations (2.1). If the condition

$$\operatorname{Re} C_\alpha^* C_\beta^* \leqslant 0, \ \operatorname{Im} C_\alpha^* C_\beta^* = 0 \tag{2.2}$$

is not satisfied, the trivial solution of (2.1) is Liapunov unstable. If condition (2.2) is satisfied, stability is ensured for the system reduced to cubic terms.

Theorem 2. Let the resonance interaction $\lambda_\alpha = \lambda_\beta = 2\lambda_\gamma$ be present in system (2.1). Let us consider the terms $C_\alpha^* z_\gamma^2$, $C_\beta^* z_\gamma^2$, $(C_\gamma^* \to \alpha) z_\alpha \bar{z}_\gamma$, and $(C_\gamma^* \to \beta) z_\beta \bar{z}$ in the right-hand sides of respective equations of system (2.1), and compose the expression $C^* = C_\alpha^* (C_\gamma^* \to \alpha) + C_\beta^* (C_\gamma^* \to \beta)$. Then, if conditions

$$\operatorname{Re} C^* \leqslant 0, \ \operatorname{Im} C^* = 0 \tag{2.3}$$

is not satisfied, the trivial solution of (2.1) is Liapunov unstable. If condition (2.3) is satisfied, stability of the system reduced to cubic terms is ensured, provided that $C^* \neq 0$.

Here $(C_\gamma^* \to \varkappa)$ is the coefficient C_γ^* of the equation in z_γ of system (2.1) for $z_\varkappa \bar{z}_\gamma$.

Proof of these statements is omitted here, because Theorems 1 and 2 are modifications of theorems in [4 –6] which bring them to the required form. We would only point out that the presence of the zero root which corresponds to the first of Eq. (2.1) does not affect the reasoning about Liapunov instability and stability of the reduced system when $\operatorname{Re} A_{ss} = 0$ $(s = 1, \ldots, 4)$.

In the presence of resonance of the form $\lambda_\gamma = \lambda_\alpha + \lambda_\beta$ with condition $\operatorname{Re} A_{ss} = 0$ $(s = 1, \ldots, 4)$ the problem of stability is determined, as in [5 –7], by coefficients of the following terms in the right-hand sides of equations of system (2.1): $C_\alpha^* \bar{z}_\beta z_\gamma$, $C_\beta^* \bar{z}_\alpha z_\gamma$, and $C_\gamma^* z_\alpha z_\beta$.

We begin by analyzing the case in which third order resonances defined by (1.6) are absent. Substitutions and transformations omitted here owing to their unweildiness yield Eqs. (2.1) in explicit form. Then, carrying out the polynomial transformation to new complex conjugate variables u_j and v_j $(j = 1, \ldots, 4)$ (with z_s and $\bar{z}_s$ in the form of second order polynomials of η_*, u_j, v_j) [8], we finally obtain

$$\eta_*^{\cdot} = 0 \tag{2.4}$$

$$u_1^{\cdot} = i\lambda_1 u_1 + \frac{n-1}{n+3} i u_1 \eta_* + \ldots, \qquad u_2^{\cdot} = i\lambda_2 u_2 + \frac{n-1}{\sqrt{n+3}} i u_2 \eta_* + \ldots$$

$$u_3^{\cdot} = i\lambda_3 u_3 - \frac{n-1}{n+3} K_3 i u_3 \eta_* + \ldots,$$

$$u_4^{\cdot} = i\lambda_4 u_4 + \frac{n-1}{n+3} K_4 i u_4 \eta_* + \ldots$$

$$K_j = \{(1-n)(1-\mu)[(1-n)^2 \mu'^2 + 4\lambda_j^2] - [2(1-n)^2 \mu'^2 + \lambda_j^2] \times L_j + (1-n)\mu L_j^2\} [\lambda_j L_j (\lambda_3^2 - \lambda_4^2)]^{-1} \qquad (j = 3, 4)$$

where the omitted terms are of an order not lower than the third with respect to η_*, u_j, v_j $(j = 1, \ldots, 4)$, and differentiation is carried out with respect to

$d\theta = \omega dt$. The complex conjugate system and its solutions are omitted here and below for the sake of brevity.

The system derived from (2.4) by the rejection of terms of order higher than the second with respect to variables is readily integrable. We have

$$\begin{aligned}
\eta_* &= c \\
u_1 &= c_1 \exp\left\{\left(\lambda_1 + c\frac{n-1}{n+3}\right) i\omega t\right\} \\
u_2 &= c_2 \exp\left\{\left(\lambda_2 + c\frac{n-1}{\sqrt{n+3}}\right) i\omega t\right\} \\
u_3 &= c_3 \exp\left\{\left(\lambda_3 - cK_3\frac{n-1}{n+3}\right) i\omega t\right\} \\
u_4 &= c_4 \exp\left\{\left(\lambda_4 + cK_4\frac{n-1}{n+3}\right) i\omega t\right\}
\end{aligned} \tag{2.5}$$

where c is a real and c_j and $\bar{c}_j$ $(j = 1, \ldots, 4)$ are complex conjugate constants of integration.

Let us formulate the obtained result.

Theorem 3. Constant Laplace solutions of the three-body problem retain their stability in the second order throughout the region of the necessary conditions of the Routh–Joukowski stability only when there are no resonances (1.6), and the solution of system (2.1) reduced to its cubic terms is determined by second order polynomials of periodic functions (2.5).

Let us now consider the case when resonances (1.6) are present.

Analysis of the structure of quadratic terms of system (2.1) for resonances $\lambda_1 = \lambda_3 \pm \lambda_4$ shows that in this problem they vanish (degenerate resonance). In fact, expressions for Φ_2 and Φ_3 do not contain terms linear with respect to Ω_1 and Ω_2, while Φ_4 and Φ_5 are linear with respect to Ω_1 and Ω_2.

Resonances $\lambda_1 = 2\lambda_2$, $\lambda_1 = 2\lambda_3$, and $\lambda_1 = 2\lambda_4$ are similarly degenerate.

The calculation of resonance coefficients for the resonance $\lambda_4 = 2\lambda_1$ yields

$$C_1^* = \left(1 - \frac{i}{\sqrt{3}}\right)\frac{(1-n)\mu' + (2\lambda_4 - L_4)i}{L_4}$$

$$C_4^* = \frac{3}{16}\left(1 + \frac{i}{\sqrt{3}}\right)\frac{(1-n)\mu' - (2\lambda_4 - L_4)i}{\lambda_3^2 - \lambda_4^2}$$

Taking into consideration that in the case of resonance $\lambda_4 = 2\lambda_1$ we have $\lambda_1 = 1$, $\lambda_4 = 2$, $\lambda_3 = n - 3$, $n \geqslant 17$, we obtain $C_1^*C_4^* = (1-n)/[(n-3)(n+1)] < 0$, which shows that condition (2.2) of Theorem 1 is satisfied.

The test of conditions of Theorem 1 for the remaining two resonances $\lambda_2 = 2\lambda_4$, and $\lambda_3 = 2\lambda_4$ requires such bulky calculations that they can only be carried out on a computer. This was done in [2] in the case of $\eta_* = 0$. Theorem 1 shows that the conclusions reached in [2] hold also for $\eta_* \neq 0$.

As indicated above, the interaction of resonances $\lambda_1 = \lambda_2 = 2\lambda_4$ occurs when $n = -2$ and $\nu = 1/36$, with the degenerate resonance $\lambda_1 = 2\lambda_4$ and strong resonance $\lambda_2 = 2\lambda_4$ [2] (in the terminology of [9]). In other words condition (2.3) of Theorem 2 is violated, which means that the interaction of resonances $\lambda_1 = \lambda_2 = 2\lambda_4$ results in instability.

Let us summarize the results of the above analysis and of that in [2].

T h e o r e m 4. The constant Laplace solutions of the unrestricted three-body problem are Liapunov unstable, if the index n and the masses of the three bodies satisfy the relations

$$-3 < n \leqslant -\frac{7}{9}, \quad \nu = \frac{16}{75}\left(\frac{n+3}{n-1}\right)^2;$$
$$-3 < n \leqslant 13 - 8\sqrt{3}, \quad \nu = \frac{1}{4}\left(\frac{n+3}{n-1}\right)^2$$

For the remaining values of masses and of index n that satisfy relations (1.6) the Laplace solutions retain their stability in the second order.

Note that Theorems 3 and 4 completely resolve the question of stability of Laplace solutions in the second order.

In the particular case when one of the masses is negligibly small in comparison with the other two and $n = -2$, the detected instability confirms the conclusion reached in [10] about the instability of triangular libration points of the restricted three-body problem with $m_1 = 0.024294 \ldots$.

The author thanks V. V. Rumiantsev and L. A. Kunitsyn for their interest in this paper.

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Translated by J. J. D.